EC 3210 Solutions

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Assignment 4

3.5. Calculate the Brewster angle for light passing from air into
a. ... GaAs ($n = 3.5$)?
b. ... plastic ($n = 1.40$)?
Repeat the calculations for the *critical angle* for light passing from the material into air.
We want the Brewster angle ...

a. ... from air into GaAs ($n = 3.5$).

From air ($n_1 = 1.0$) into GaAs ($n_2 = 3.5$):

$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}\left(\frac{3.5}{1}\right) = 75.1^\circ\,. \tag{1a}$$

From GaAs ($n_1 = 3.5$) into air ($n_2 = 1.0$):

$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}\left(\frac{1}{3.5}\right) = 15.95^\circ\,. \tag{1b}$$

Note: You should be able to prove that, assuming parallel surfaces, light incident on an air-material interface at the Brewster angle will also be at the Brewster angle at the material-air interface. (See Prob. 3.7.)

b. ... air into plastic ($n = 1.40$).

From air ($n_1 = 1.0$) into plastic ($n_2 = 1.4$):

$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}\left(\frac{1.4}{1}\right) = 54.5^\circ\,. \tag{2a}$$

From plastic ($n_1 = 1.4$) into air ($n_2 = 1.0$):

$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}\left(\frac{1}{1.4}\right) = 35.5^\circ\,. \tag{2b}$$

We now must do the calculations for the critical angle when going from the material into the air. (Note that there is no critical angle when going from the air into the material.)

a. The critical angle going from GaAs into air is

$$\theta_c = \sin^{-1}\left(\frac{n_1}{n_2}\right) = \sin^{-1}\left(\frac{1}{3.5}\right) = 16.6^\circ\,. \tag{3}$$

b. The critical angle going from plastic into air is

$$\theta_c = \sin^{-1}\left(\frac{n_1}{n_2}\right) = \sin^{-1}\left(\frac{1}{1.4}\right) = 45.6^\circ\,. \tag{4}$$

3.7. Consider a piece of glass with parallel faces spaced a distance d apart placed in the path of a plane-wave light beam. The angle of incidence at the air-glass interface is the Brewster's angle of that interface. Show that the angle of incidence at the glass-air interface is equal to the glass-air Brewster's angle.

A plane wave enters a glass block of thickness d from air at the Brewster angle. We want to show that the angle of incidence at the glass-air interface is also at the Brewster angle for that interface (assuming that the sides of the block are parallel). See Fig. 1 for the geometry.

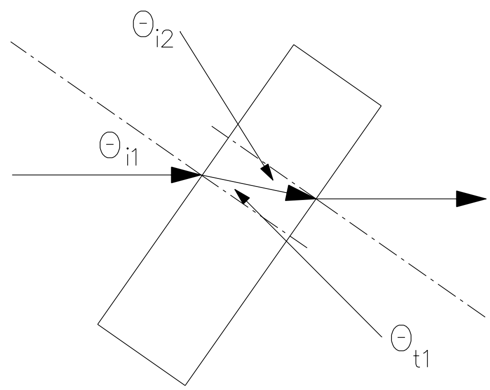

Figure 1: Geometry of Problem 3.7.

The relation for the angle of incidence θ_{i1} at the air-glass interface is

$$\theta_{i1} = \theta_{B1} = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}\left(\frac{n_2}{1}\right)\,. \tag{5}$$

We can draw a right triangle having this relationship, as shown in Fig. 2.

We now need to compute the transmission angle θ_{t1} at the air-glass interface. We use Snell's Law for this.

$$n_1 \sin\theta_{i1} = n_2 \sin\theta_{t1} \tag{6a}$$

$$1 \times \frac{n_2}{\sqrt{n_2^2+1}} = n_2 \sin\theta_{t1} \tag{6b}$$

$$\sin\theta_{t1} = \frac{1}{\sqrt{n_2^2+1}} = \sin\phi\,, \tag{6c}$$

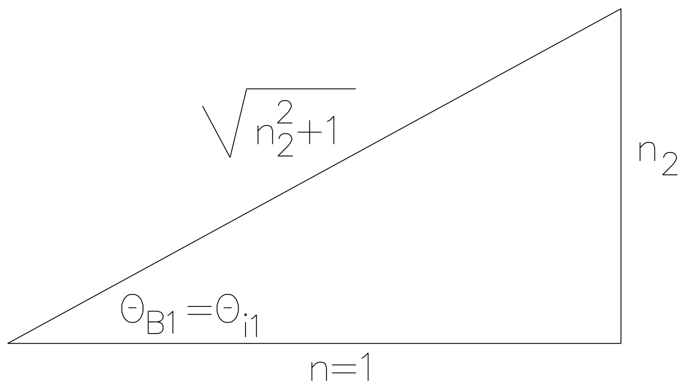

Figure 2: Trigonometry relations for Prob. 3.7.

where ϕ is defined in Fig. 2.

From the geometry of Fig. 1 we note that $\theta_{i2} = \theta_{t1}$ by similar triangles.

$$\theta_{i2} = \theta_{t1} = \sin^{-1}\left(\frac{1}{\sqrt{n_2^2+1}}\right) = \tan^{-1}\left(\frac{1}{n_2}\right) = \theta_{B2}\,. \tag{7}$$

3.8. Consider the following cases. Suppose that we have a quarter-wave plate with fast axis (y') that is at 45° from the horizontal. Write the phasor representation of the input wave, compute the phase shift of the quarter-wave plate, and compute the phasor at the rear surface of the quarter-wave plate for the following cases. Is the output linearly polarized or circularly polarized? In which direction? The cases are ...

a. ... a horizontally polarized wave incident on the quarter-wave plate,

b. ... a right-circular polarized wave incident on the quarter-wave plate, and

c. ... a left-circular polarized wave incident on the quarter-wave plate.

We are given a $\lambda/4$-plate with its fast axis at a 45° angle off the horizontal (see Fig. 3). We will call the fast axis the y' axis and the slow axis will be the x' axis. (This will ensure a right-handed coordinate system.)

For a $\lambda/4$-plate the fast-axis wave (the y' wave) will receive an additional $\pi/2$ phase shift above that incurred by the slow-axis wave (the x' wave).

a. The input wave is horizontally polarized. Assuming that the instantaneous E-field points in the rightward direction along the horizontal axis, the phasor representation *at the input* is:

$$\tilde{E}_{x'} = -0.707E_0e^{j0} = 0.707E_oe^{+j\pi} \tag{8a}$$

$$\tilde{E}_{y'} = 0.707E_oe^{j0}\,. \tag{8b}$$

(Note that the x$'$ component is in the $-$x$'$-direction.)

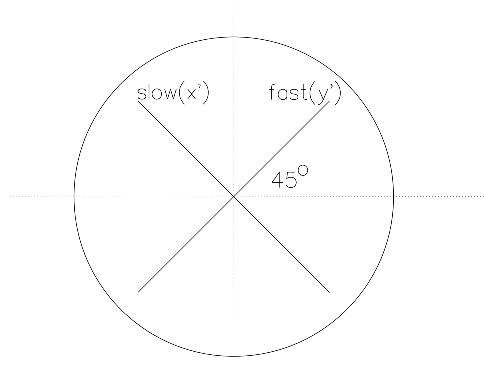

Figure 3: Geometry of fast and slow axes of quarter-wave plate.

The phasor representation *at the output* is:

$$\tilde{E}_{x'} = 0.707E_o e^{j\pi} \tag{9a}$$

$$\tilde{E}_{y'} = 0.707E_o e^{j0} e^{+j\pi/2} = 0.707E_o e^{+j\pi/2}\,. \tag{9b}$$

(Here we have added an extra $\pi/2$ to the phase of the light oriented along the fast axis to represent the relative phase sift in the quarterwave plate.)

We have equal amplitudes and a relative phase difference of $\Delta\phi = \phi_{\text{fast}} - \phi_{\text{slow}} = \phi_{y'} - \phi_{x'} = (\pi/2) - \pi = -\pi/2$. We see that we have equal amplitudes and a phase difference of $-\pi/2$, so we conclude that we have circular polarization at the output of the waveplate.

For the purpose of determining if the light is left or right polarized, we need to find $\phi_{x'} - \phi_{y'} = +\pi - (\pi/2) = +\pi/2$. Based on our class discussion, this is **LCP (left circularly polarized) light**.

b. The input wave is RCP. This means that the light has equal amplitudes and a phase shift of $-\pi/2$ between the fast-axis and slow-axis components, i.e., $|\tilde{E}_{x'}| = |\tilde{E}_{y'}| = E_0$ and that $\Delta\phi = \phi_{x'} - \phi_{y'} = -\pi/2$.

Hence, a phasor representation *at the input* is:

$$\tilde{E}_{x'} = E_o e^{j0} \tag{10a}$$

$$\tilde{E}_{y'} = E_o e^{+j\pi/2}\,. \tag{10b}$$

At the output of the quarterwave plate the light polarized along the fast axis (i,e., the y'-axis)will have $+\pi/2$ more phase than the slow axis (i.e., the x'-axis. Hence, a phasor representation *at the output* is:

$$\tilde{E}_{x'} = E_o e^{j0} \tag{11a}$$

$$\tilde{E}_{y'} = E_o e^{+j\pi/2} e^{+j\pi/2} = E_0 e^{+j\pi} = -E_0\,. \tag{11b}$$

We have equal amplitudes and a relative phase difference of $\Delta\phi = \phi_{x'} - \phi_{y'} = 0 - \pi = -\pi$. The components have equal amplitudes and have a $-\pi$ phase difference; this is linearly polarized light. The vector sum of the components (illustrated in Fig. 4) produces **horizontally polarized light** (i.e., linearly polarized light in the horizontal plane.

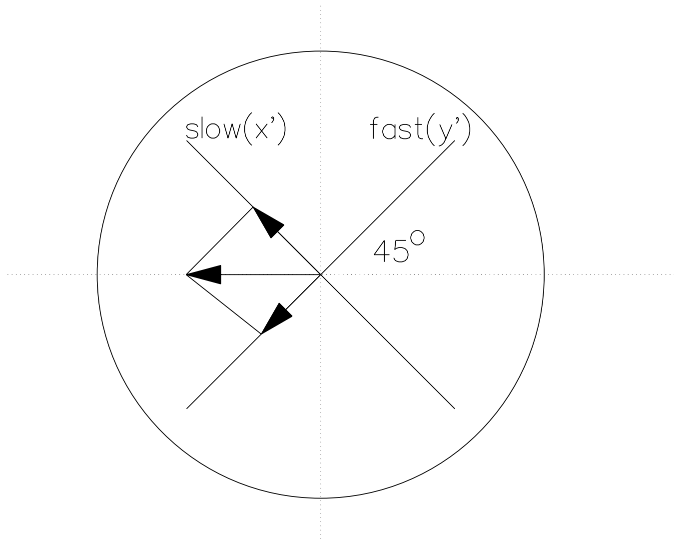

Figure 4: Reconstruction from components (Prob. 3.8b).

c. The input wave is LCP (i.e., left circular polarized). This means that $|\tilde{E}_{x'}| = |\tilde{E}_{y'}| = E_0$ and that $\Delta\phi = \phi_{x'} - \phi_{y'} = +\pi/2$.
A phasor representation *at the input* is:

$$\tilde{E}_{x'} = E_o e^{+j\pi/2} \tag{12a}$$
$$\tilde{E}_{y'} = E_o e^{j0} \,. \tag{12b}$$

A phasor representation *at the output* of the quarterwave plate is:

$$\tilde{E}_{x'} = E_o e^{+j\pi/2} \Rightarrow E_0 e^{j0} \tag{13a}$$
$$\tilde{E}_{y'} = E_o e^{j0} e^{+j\pi/2} = E_0 e^{+j\pi/2} \Rightarrow E_0 e^{j0} \,. \tag{13b}$$

We have equal amplitudes and a relative phase difference of $\Delta\phi = \phi_{x'} - \phi_{y'} = 0$. The wave are equal amplitude and have a 0 phase difference; this is linearly polarized light. The vector sum of the components (illustrated in Fig. 5) produces **vertically polarized light**.

3.9. Consider linearly polarized light incident on a half-wave plate. The angle between the fast axis of the plate and the polarization axis of the light is $+\theta$ (positive angle is counter clockwise). Prove that the angle between the fast axis and the polarization axis of the light at the output is $-\theta$ (i.e., that the polarization axis has been rotated by a total angle of 2θ from its input orientation with a rotation direction that is through the fast axis of the wave plate.

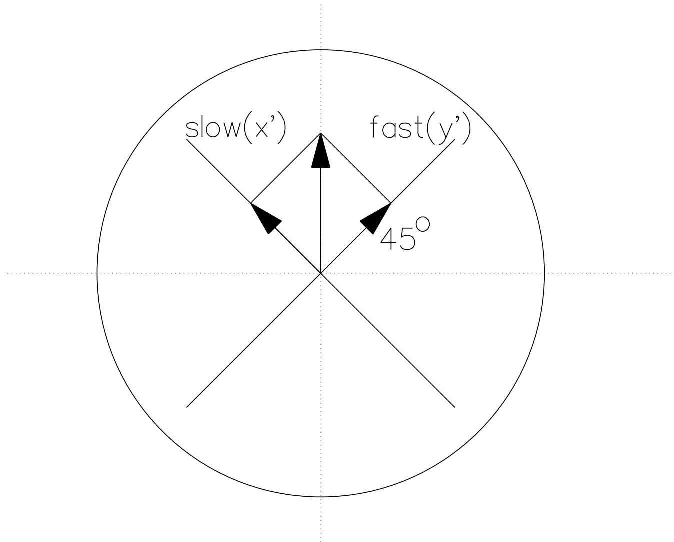

Figure 5: Reconstruction from components (Prob. 3.8c).

We are given a half-wave plate; the fast-axis wave (the y'-axis wave) receives an additional phase shift of π beyond that of the slow-axis wave (the x'-axis).

We are also given a linearly polarized input wave that makes an angle θ with the fast axis of the wave plate (see Fig. 6). We assume (arbitrarily) that the fast axis of the wave plate is vertical.

A phasor representation *at the input* is:

$$\tilde{E}_{\text{fast}} = E_o \cos\theta e^{j0} \tag{14a}$$

$$\tilde{E}_{\text{slow}} = E_o \sin\theta e^{j0} \,. \tag{14b}$$

The half-waveplate causes the fast axis to have a phase shift that is π bigger than the phase shift of the slow axis. A phasor representation *at the output* is:

$$\tilde{E}_{\text{fast}} = E_o \cos\theta e^{j0} e^{j\pi} = -E_0 \cos\theta \tag{15a}$$

$$\tilde{E}_{\text{slow}} = E_o \sin\theta e^{j0} = E_0 \sin\theta \,. \tag{15b}$$

We have unequal amplitudes and a relative phase difference of $\Delta\phi = \phi_{\text{fast}} - \phi_{\text{slow}} = \pi$. This is linearly polarized light. (The fast-axis component at the output is flipped 180° from its original orientation; the slow-axis component retains its original orientation.) The vector sum of the components (illustrated in Fig. 7) produces linearly polarized light in the plane shown. This plane is located an angle θ on the opposite side of the fast axis from the plane of polarization of the input wave.

3.10. Draw the orientation of the fast axis of a half-wave plate to convert vertical linear polarization into horizontal linear polarization.

To rotate vertically polarized light to horizontally polarized light, we want the fast axis of the half-wave plate to be at 45° from vertical as shown in Fig. 8.

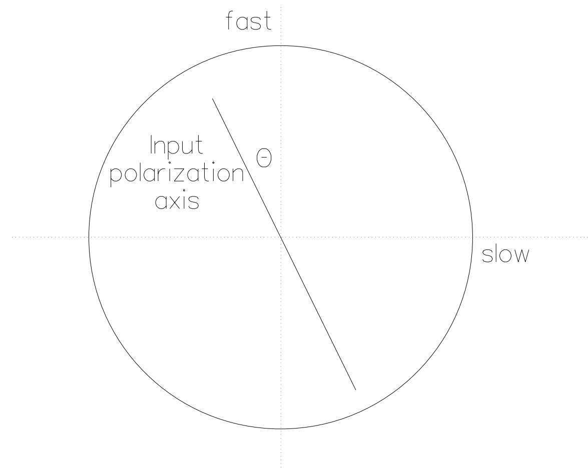

Figure 6: Geometry of input polarization (Prob. 3.9) (as seen from the wave).

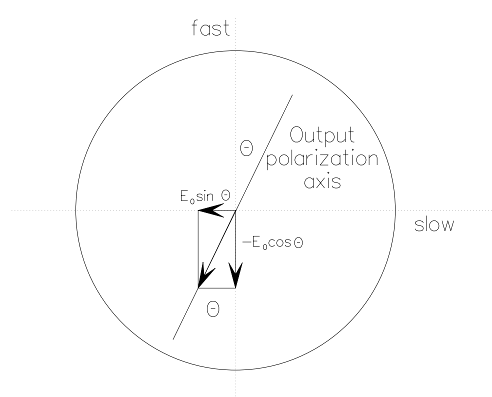

Figure 7: Reconstruction of polarization plane from components (Prob. 3.9) (as seen from the wave).

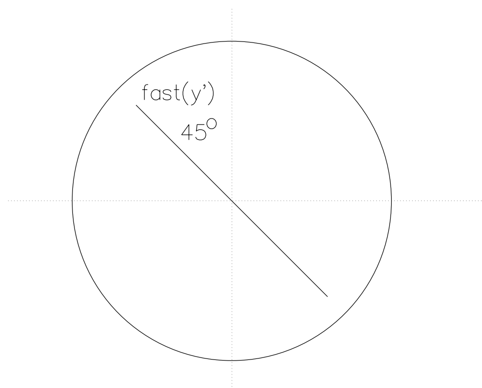

Figure 8: Orientation of fast axis (Prob. 3.10) as seen from the front of the waveplate.